TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 503

EFFECT OF STRESSED COVERING ON STRENGTH OF INTERNAL GIRDERS OF A WING

By H. Tellers

From the 1928 Yearbook of the
Wissenschaftliche Gesellschaft für Luftfahrt

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The problem of calculating the bending stress in an ordinary straight beam is statically undetermined, since nothing is known as yet regarding the distribution of the stress over the cross section. This question can be definitely explained only by the elastic deformation. In order to clear away this uncertainty, the simple girder theory assumes that, even after deformation, the sections perpendicular to the girder axis remain flat. This hypothesis, which was first arbitrarily made by Bernoulli, is confirmed to a certain degree, especially by the fact that the conclusions derived from it agreed well with experience, particularly as regards bodies following Hooke's law. In connection with this law, it follows that normal stresses are transmitted according to the well-known rectilinear law in flat sections. Subsequently, De St. Venant's strictly theoretical investigation proved that the linear law necessarily applies to bodies following Hooke's law, provided the fibers parallel to the girder axis exert no mutual transverse or tensile stresses and no shearing stresses transversely to the girder axis. These conditions

*"Ueber die Mittragende Breite," from the 1928 Yearbook of the Wissenschaftliche Gesellschaft für Luftfahrt, pp. 100-104.

are fulfilled accurately enough for girders, the cross section of which does not materially differ from a rectangle, and hence, for such girders, there is a good agreement between experience and the single girder theory.

If Bernoulli's hypothesis were applied to girders, the stress along the width of the flange would be constant. This, however, is far from being the case for girders with wide flanges, where the stress decreases toward the outer edge of the girder flange (Fig. 1). With approximation, such a girder can be considered as a disk stressed at the junction point of web and flange by the deformation of the web fibers. It is obvious that the fibers near the edge are less affected than those near the junction point.

In practice the actual maximum stress is greater than the stress determined by the simple girder theory, which overestimates the bearing or supporting capacity of the flange. The fact is that the assumptions of the simple girder theory no longer hold true, since normal transverse and shearing stresses are engendered in the plane of the flange.

We can assume the true width of the flange b to be replaced by a smaller width λ, along which the stress is considered constant and equal to its value at the extreme web fiber. The magnitude of λ should be so determined that the calculated bending strength of the aforesaid girder with a flange width λ would equal the bending strength of the actual girder with a width b. Hence it can be said that only the width λ possesses full bearing capacity. The problem of determining the "fully supporting" width is frequently encountered in practice, e.g., in bulkheads, tank walls, stiffened ceilings and, furthermore, in airplane construction, etc. (Figs. 2 and 3).

The importance of this problem had long been recognized and attempts were made to overcome the difficulties by rule-of-thumb formulas which, however, lacked accuracy and general applicability.

In his work (contained in the "Beiträge zur technischen Mechanik und technischen Physik," published by Springer in 1924) Von Karman indicated a method for the determination of the actual stress distribution or of the supporting width. In this work the method which can be used for any girder arrangement and any load distribution was developed for a continuous girder resting on an infinite number of equidistant supports (spacing of supports = $2\,l$), the same load being symmetrically applied in the center of each span. The bending moment is represented by a trigonometrical series which, on account of the symmetrical load, has the form:

$$M = M_0 + M_1 \cos \frac{\pi x}{l} + M_2 \cos \frac{2\pi x}{l} + \ldots\ldots \qquad (1)$$

The flange of uniform thickness δ is considered as a disk, and its bending strength as a plate is neglected. The stresses in the flat disk are indicated by the well-known Airy stress function F. This function must satisfy the marginal conditions. For $x = 0$ and $x = 2l$ (Fig. 1):

a) The angular variation in the horizontal plane = 0.

b) The displacement in the x direction = 0 (for reasons of symmetry).

Moreover, the displacement in the y direction is assumed to disappear along $y = 0$. As a matter of fact, this does not actually hold true, on account of the transverse contraction at the junction point, but it will be shown subsequently that this approximation agrees well with the test results.

For a finite flange width, we must also have, at the free edge $y = b$,

$$\sigma_y = 0 \quad \text{and} \quad \tau = 0^*$$

These boundary conditions can be expressed by the stress function F. Consequently, a function F is sought, satisfying the indicated boundary conditions, as well as the differential equation of the plane disk $\Delta \Delta F = 0$. Besides, the function must be so general as to cover any stress distribution which may be set up in the disk by shearing stresses acting symmetrically to the center of the span at the junction point, but otherwise arbitrarily distributed about this point.

*In this connection, attention is called to a dissertation by Mr. Metzer, which is to appear shortly and will contain, in connection with Von Karman's work, the development of a few practically important cases.

This last condition is satisfied by the expression

$$F = \sum_{n=1}^{\infty} f_n(y) \cos \frac{n\pi x}{l} \tag{2}$$

which, on account of the cosine member, is symmetrical when x = l. When this expression is introduced in the equation $\Delta \Delta F = 0$, a determinative equation of $f_n(y)$ is obtained. Its solution reads as follows:

$$f_n(y) = A_n e^{\alpha_n y} + B_n e^{-\alpha_n y} + C_n y e^{\alpha_n y} + D_n y e^{-\alpha_n y} \tag{3}$$

when $\alpha_n = \frac{n\pi}{l}$

Thus the above expression of the Airy stress function changes to

$$F = \sum_{n=1}^{\infty} [A_n e^{\alpha_n y} + B_n e^{-\alpha_n y} + C_n y e^{\alpha_n y} + D_n y e^{-\alpha_n y}] \cos(\alpha_n x) \tag{4}$$

For an infinite plate width, the members in $e^{\alpha_n y}$ vanish, if all the stresses are of finite magnitude for $y = \infty$. Function F is given by equation (4) for a finite plate width. By means of the marginal conditions the constant values B_n, C_n, and D_n can be expressed by A_n at the edges $y = 0$ and $y = b$, so that the stress function adapted to the marginal conditions assumes the following form:

$$F = \sum_{n=1}^{\infty} A_n \varphi_n(y) \cos \frac{n\pi x}{l} \tag{5}$$

where $\varphi_n(y)$ is a given function of y.

The flange can be considered as a constant elastic support of the web. Under these conditions the web would be considered as a statically undetermined girder, in which the constant supporting action is exerted by the shearing stresses at the junction point. Hence the shearing stresses must be considered as statically undetermined values. When they are again replaced by the stress function F (equation 5), the coefficients A_n are to be considered as statically undetermined.

For a given loading, the coefficients M_1, M_2, etc., of the bending moment developed in a trigonometrical series are known, whereas M_o is undetermined and has therefore to be considered as the statically undetermined value.

A_n and M_o are determined according to the principle of the least work of deformation which involves the work done by the flanges and webs. The work of deformation is expressed by the stresses, and the stress function F is introduced, whereupon, after satisfying the conditions of equilibrium, the total work of deformation appears as a function of the statically undetermined quantities M_o and A_n. These quantities must be given such values as to reduce the total work of deformation to a minimum. Hence the partial differential quotients derived from the statically undetermined quantities M_o and A_n must be equal to zero, whence $M_o = 0$. Furthermore, we obtain a determinative equation for A_n, which depends on the parameter n and also on Poisson's constant of the material m, on constructional data and on the loading

M_n. M_n can be determined for a given loading, whence A_n is likewise given. The stresses in the flanges are defined by the second differential coefficients of the stress function F, and thus all the data required for the determination of the stresses in the continuous girder are known.

Let us consider more closely the case of an identical "point load" P in the center of each span. The corresponding moment diagram is shown in Figure 4.

The moment can be represented as a straight periodic function of the coordinate x with a period of $2l$. The coefficients M_n of the linear function developed in a pure cosine series can be easily calculated for the moment. We obtain

$$M_n = \frac{2\,P\,l}{\pi^2}\;\frac{1}{n^2}$$

Figure 4.- The central part l of the girder with a span $2l$ has the same moment diagram as a girder on free supports of a span l with free ends. Owing to this fact, one may be led to believe that the supporting width and the stress diagram calculated for the central girder portion resting on an infinote number of supports (span $2l$) directly apply to a girder with free ends, provided the span is l. This is likewise confirmed by Metzer, who indicates the relation for the bearing or supporting width of both plate girders and states that the aforesaid girder with free ends and a span l is identical with the central portion of the continuous girder

under consideration. Of course I should not fail to mention that, in the investigation of the girder with the free ends, not all the marginal conditions were strictly satisfied. Thus, for instance, the question was simplified by passing over the conditions according to which the shearing stress of the flanges would vanish at the free ends. On the other hand, it was shown by a simple example that the influences of the residual shearing stresses is small at the flange tips of the portion of a length l cut out of the continuous girder. Altogether the central girder portion between the moment zero points of the continuous girder can be used for the calculation of the stress diagram or of the bearing width of a girder placed on two supports and having free ends. Owing to the symmetry the calculation can be confined to the portion between $x = 0$ and $x = \frac{l}{2}$.

Stress and elongation measurements were then carried out for a steel-flange girder resting on two supports (span l). The calculation was made according to the aforesaid method for a continuous girder having the same section but twice the span (Mr. Miller's dissertation, which will soon be published). I shall now show you a few figures in which the calculated and the measured values of $E_x E$ are compared (Figs. 5-8). There is a good agreement throughout between the calculated and the measured $E_x E$ values of the flanges. The linear stress distribution in the web is likewise confirmed by the test. This fact was assumed for the theoretical consideration. A certain discrepancy is found in the section above the point of application of the load, but it is probably caused by the local force transmission (external force). The relatively greatest deflection is near the junction point, which may be explained as follows.

According to the theory (Fig. 9), there is a small discontinuity in the stress diagram near the junction point due to the fact that a free transverse contraction was assumed for the web, the transverse stress being equal to zero, which, however, does not hold good for the plate. The difference between the stress σ_{xg} of the flange and the stress σ_{xs} of the web at the junction point is approximately 6%. This slight inaccuracy had already been designated by Von Karman as a negligible defect.

I shall now show you some other results computed from Mr. Metzer's dissertation (Figs. 10-14) "Discussion of the Results."

It appears from these results that, beginning with a certain value of $\frac{b}{l}$, the bearing width does not increase materially. It is therefore of no use to increase the flange width beyond this value, since there will be no further appreciable increase in λ. It will be shown subsequently that it is not only useless to increase the flange width beyond a certain value, but may even be detrimental.

The results just shown refer to girders with stiff flanges. The conditions arising for very thin flexible flanges cannot be anticipated. As shown in Figure 15, the flange is then bent not only in the longitudinal direction but also transversely, which causes a deformation of the section outline. It may be noted that this deformation not only occurs for compression stresses of the flange which characterize buckling phenomena, but that the deformations are much greater when the flange is subjected to tensile stresses. For an appreciable increase in these transverse deflections, the assumptions of the considered method will no longer hold good. The question of transverse deflection has not yet been sufficiently explained. Since it affects the bearing width, it is well in such cases to determine the carrying or supporting width experimentally.

Next come the stress and elongation measurements of web and flange. However, one can imagine cases in which certain reasons might render stress measurements impossible. This is the case of very thin girders or coverings in which large bulges or bucklings occur. We shall therefore attempt to determine the bearing capacity by another method.

For the same load the deflection f_s of the web without flange must obviously be greater than the actual deflection f_e of the whole girder with flanges. Hence the difference between the deflections $f_s - f_e$ is a measure of the bearing width. We again assume the actual girder to be replaced by an equivalent girder with a flange width λ, so that the simple girder theory can be applied to it with the effective moment of inertia I_e. A simple consideration leads to

$$\frac{\lambda}{l} = \sigma \; \nu \frac{1}{1 - \chi \, \nu} ,$$

where σ and χ are constants given by construction.

$$\nu = \frac{J_e}{J_s} - 1$$

(I_s = moment of inertia of web alone; I_e = effective moment of inertia of whole girder, l = span.)

Furthermore, if we assume that the bearing width is approximately constant over the length of the girder, then $\frac{I_e}{I_s} = \frac{f_s}{f_e}$. Thus we might measure f_s and f_e, whereby ν becomes known and means are afforded for calculating $\frac{\lambda}{l}$. This method ought to indicate the order of magnitude of the supporting width. I shall now show you the results of the measurements which I made with duralumin I beams with thin riveted flanges (Fig. 16) "Discussion of the Results." The determination of the magnitude of the bearing width, according to this method, will be satisfactory in many cases.

We already know that the bearing width is generally variable along the beam (Fig. 11). This can best be taken care of by the following method.

Let

$$E\ J_e\left(\frac{d^2\ y}{d\ x^2}\right)_e = -\ M$$

$$E\ J_s\left(\frac{d^2\ y}{d\ x^2}\right) = -\ M$$

Hence,

$$\frac{J_e}{J_s} = \frac{\left(\frac{d^2\ y}{d\ x^2}\right)_s}{\left(\frac{d^2\ y}{d\ x^2}\right)_e}.$$

We should therefore determine $\left(\frac{d^2\ y}{d\ x^2}\right)$ (for a loading of the web without flange) and $\left(\frac{d^2\ y}{d\ x^2}\right)$ (the value for the whole beam with flanges) at different points of the beam.

According to Figure 17,

$$\frac{d\ y}{d\ x} = \text{tangent}\,\alpha$$

$$\frac{d^2\ y}{d\ x^2} = \frac{\Delta\ \text{tangent}\,\alpha}{\Delta\ x}$$

Thus Δ tangentα can be determined by fixing two mirrors to the web (Fig. 17) at a distance of Δ x. When Δ tangent α is divided by the distance Δ x between the two mirrors, we have

$$\frac{\Delta\ \text{tangent}\,\alpha}{\Delta\ x} = \frac{d^2\ y}{d\ x^2}.$$

$\frac{I_e}{I_s}$, ν and $\frac{\lambda}{l}$ become known when the above value is determined for the web alone and for the whole girder.

This method of testing affords a means of determining the bearing width, as well as the effect of the transverse deflection, even for structures with very thin flanges which cannot yet be calculated. This method furthermore enables us to determine the effect of riveting and gluing (wood) on the bearing width.

I hope to proceed with tests along this line, dealing also with series of girders, and to report later regarding the results.

In conclusion, I wish to refer to several other articles (in addition to those by Von Karman and Mr. Metzer), which deal with the problem of the bearing or supporting width. One is by Mr. Bortsch in Der Bauingenieur of 1921, and contains an attempt to determine the bearing width. A lecture by Dr. Schnadel on "Stress Distribution in the Flanges of Thin-Walled Box Girders" was published in the 1926 Yearbook of the Schiffbautechnische Gesellschaft (Society of Naval Architects).

Translation by W. L. Koporindé,
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for Aeronautics.

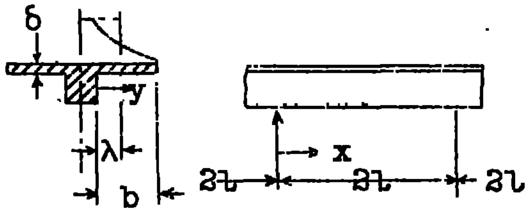
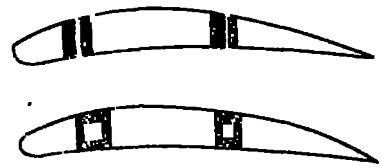


Fig.1



Figs.2 & 3

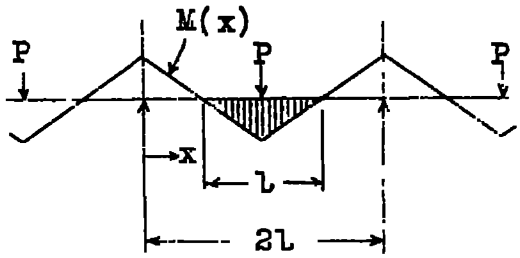


Fig.4

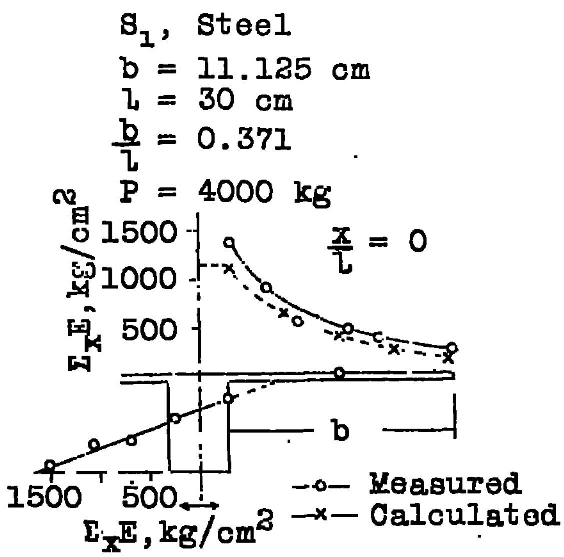


Fig.5

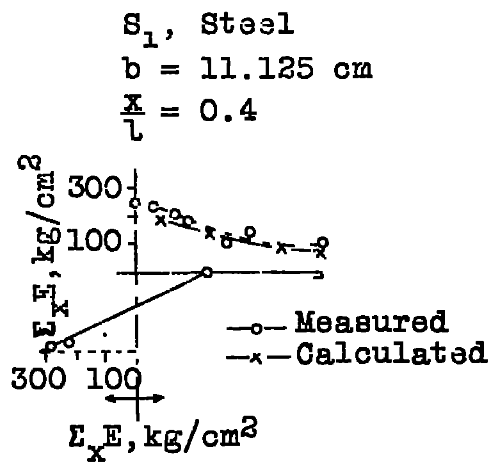


Fig.6

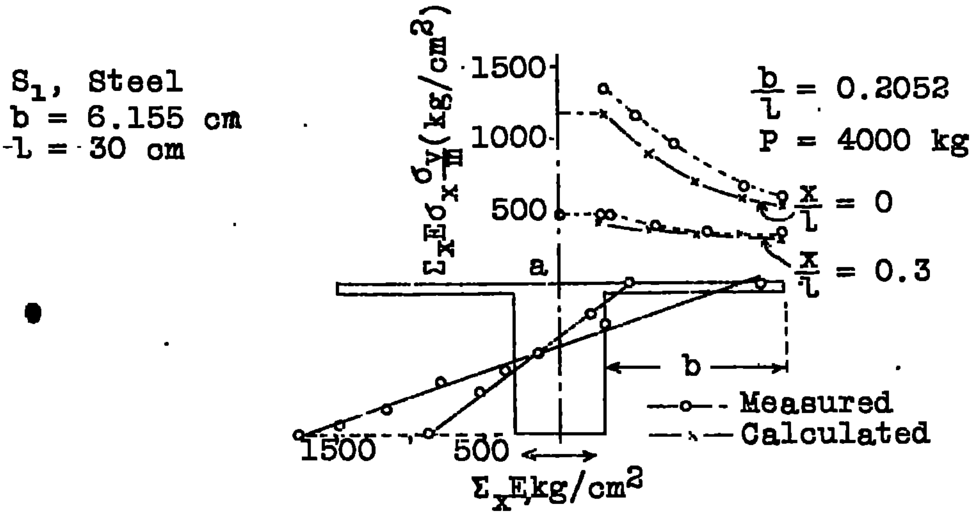


Fig.7

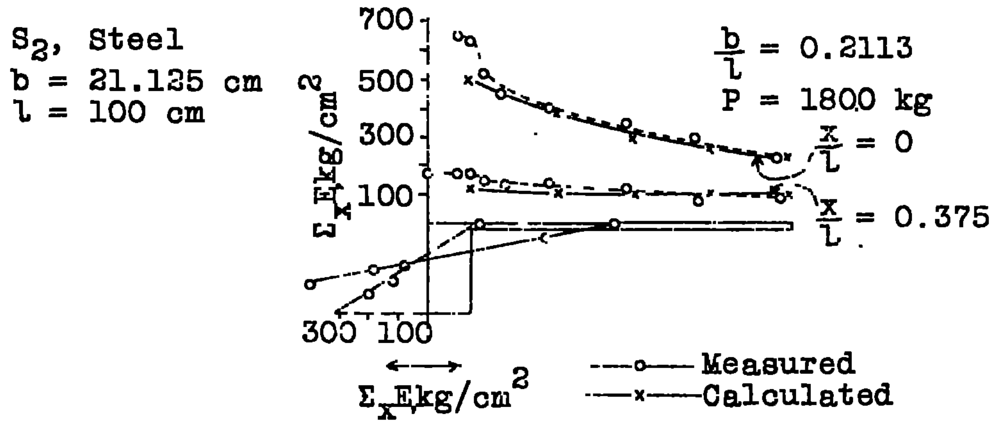


Fig.8

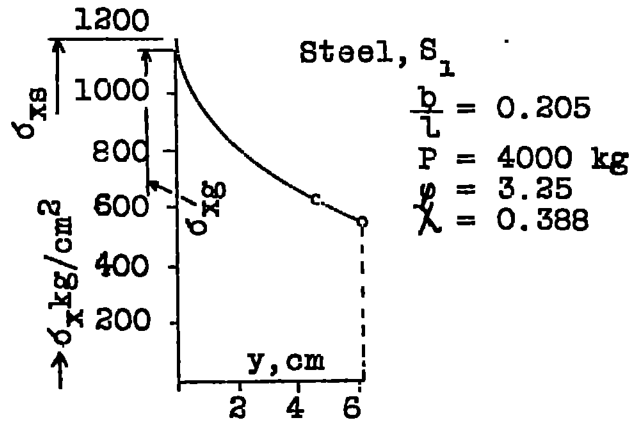
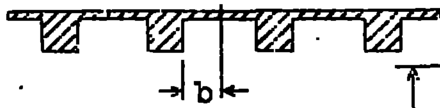


Fig.9



Bf. = Beam field, B = single girder.
d = Continuous girder on an infinite number of supports (span 2l).
f = Free support, i.e. girder resting on 2 supports (span. l), the edges remaining free.
gl.L. = Loading through uniformly distributed load.

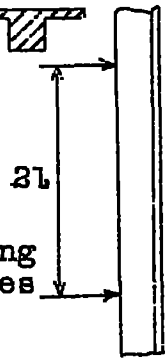


Fig.10 Bf.d.

E.L. = Loading through a point load in the center of the span.

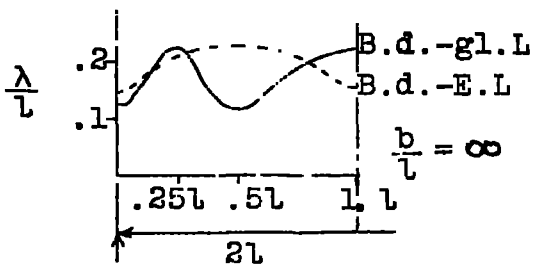


Fig.11

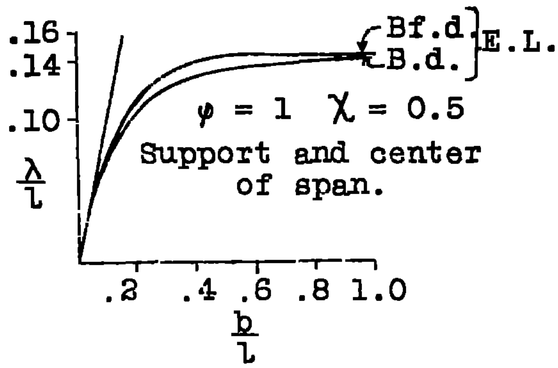


Fig.12

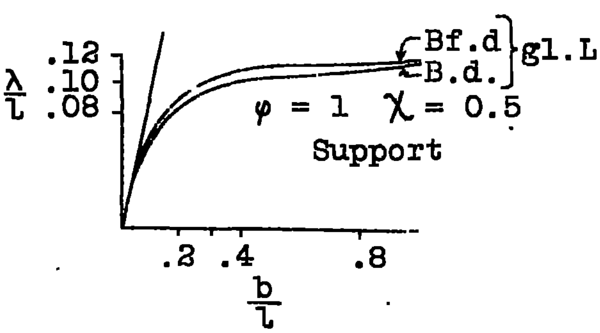


Fig.13

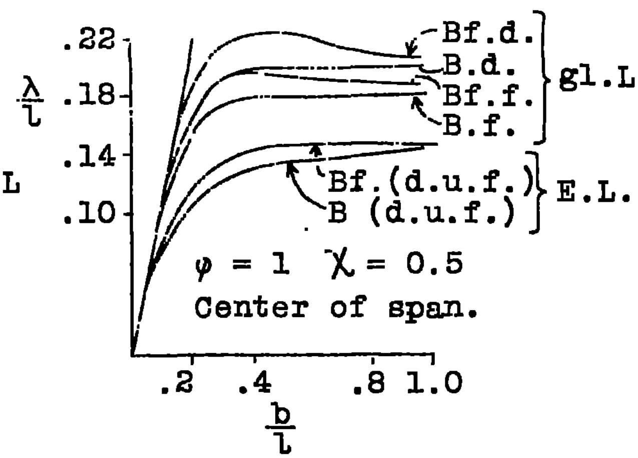


Fig.14

Fig.15

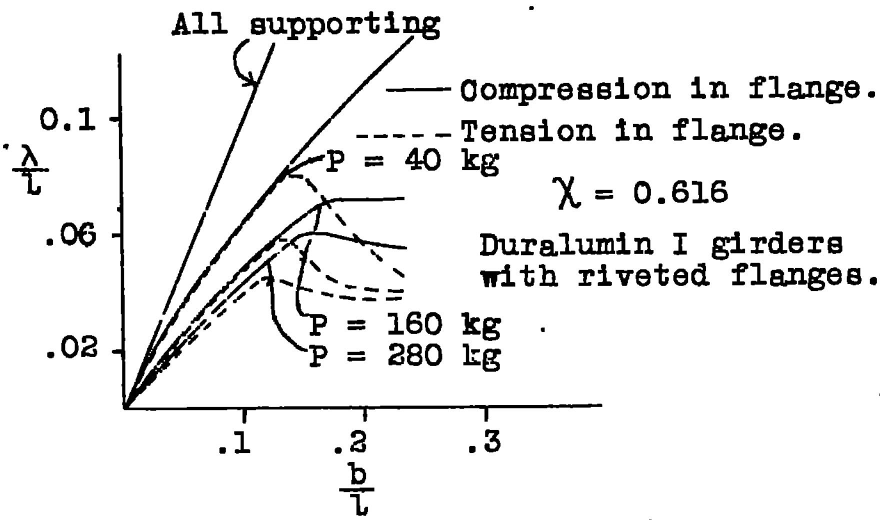
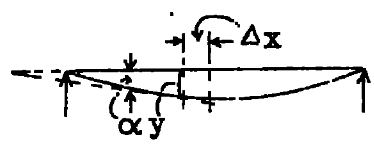


Fig.16

$\frac{dy}{dx} = \tan\alpha$



$\frac{d^2y}{dx^2} = \frac{\Delta \tan\alpha}{\Delta x}$

Fig.17